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## III. Solution by the PROPOSER.

Let  $ABCD$  be the given square whose side  $=a$ , and  $MN$  one of the straight lines within the square. Draw  $AE$  equal and parallel to  $MN$ ,  $EF$  parallel to  $AB$  cutting  $BC$  in  $F$ , and  $EG$  parallel to  $AD$  cutting  $CD$  in  $G$ . Put  $MN=AE=x$  and  $\angle EAD=\theta$ . If  $x$  and  $\theta$  were fixed the number of lines of the length  $a$  would equal the number of points in the rectangle  $EF CG$  whose area is  $a^2 - ax(\sin\theta + \cos\theta) + x^2 \sin\theta \cos\theta$ . Then if all lines are supposed to be equally distributed about the point  $M$  the required average is

$$A_1 = \frac{\int_0^{\frac{1}{2}\pi} d\theta \int_0^{a \sec \theta} [a^2 - ax(\sin\theta + \cos\theta) + x^2 \sin\theta \cos\theta] x dx}{\int_0^{\frac{1}{2}\pi} d\theta \int_0^{a \sec \theta} [a^2 - ax(\sin\theta + \cos\theta) + x^2 \sin\theta \cos\theta] dx}$$

$$= \frac{\frac{a^4}{12} \int_0^{\frac{1}{2}\pi} (2 - \tan\theta) \sec^2 \theta d\theta}{\frac{a^3}{6} \int_0^{\frac{1}{2}\pi} (3 - \tan\theta) \sec \theta d\theta} = \frac{3a}{4[\log(1+\sqrt{2})^3 + 1 - \sqrt{2}]}$$

If the lines are supposed to be so distributed as to join every possible pair of points in the square, the required average is

$$A_2 = \frac{\int_0^{\frac{1}{2}\pi} d\theta \int_0^{a \sec \theta} [a^2 - ax(\sin\theta + \cos\theta) + x^2 \sin\theta \cos\theta] x^2 dx}{\int_0^{\frac{1}{2}\pi} d\theta \int_0^{a \sec \theta} [a^2 - ax(\sin\theta + \cos\theta) + x^2 \sin\theta \cos\theta] x dx}$$

$$= \frac{\frac{a^5}{60} \int_0^{\frac{1}{2}\pi} (5 - 3\tan\theta) \sec^3 \theta d\theta}{\frac{a^4}{12} \int_0^{\frac{1}{2}\pi} (2 - \tan\theta) \sec^2 \theta d\theta} = \frac{a}{15} [2 + \sqrt{2} + 5\log(1 + \sqrt{2})]$$

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

\*Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let  $a, b, c$  be the sides;  $A, B, C$  the angles of the triangle;  $u, v$  variable lengths from  $A$  on  $b, c$ ;  $w, x$  variable lengths from  $B$  on  $a, c$ ;  $y, z$  variable lengths from  $C$  on  $a, b$ . Also let  $l_1 = \sqrt{(u^2 + v^2 - 2uv \cos A)}$ ,  $l_2 = \sqrt{(w^2 + x^2 - 2wx \cos B)}$ ,  $l_3 = \sqrt{(y^2 + z^2 - 2yz \cos C)}$ ,  $M$  = average length. Then

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\*Another solution of this problem will be published next month.

$$M = \frac{\int_0^b \int_0^c \int_0^{l_1} l \, du \, dv \, dl + \int_0^a \int_0^c \int_0^{l_2} l \, dw \, dx \, dl + \int_0^a \int_0^b \int_0^{l_3} l \, dy \, dz \, dl}{\int_0^b \int_0^c \int_0^{l_1} du \, dv \, dl + \int_0^a \int_0^c \int_0^{l_2} dw \, dx \, dl + \int_0^a \int_0^b \int_0^{l_3} dy \, dz \, dl} \\ = N/D.$$

$$\int_0^b \int_0^c \int_0^{l_1} l \, du \, dv \, dl = \frac{1}{2} \int_0^b \int_0^c (u^2 + v^2 - 2uv \cos A) \, du \, dv \\ = \frac{1}{6} \int_0^b (3u^2 c + c^3 - 3uc^2 \cos A) \, du = \frac{bc}{12} (2b^2 + 2c^2 - 3bc \cos A). \\ \therefore N = \frac{2abc(a+b+c) + a^2 b^2 \cos C + a^2 c^2 \cos B + b^2 c^2 \cos A}{12}.$$

This follows, because  $\frac{bc}{12} (2b^2 + 2c^2 - 3bc \cos A) = \frac{bc}{12} (2a^2 + bc \cos A)$ .

$$\int_0^b \int_0^c \int_0^{l_1} du \, dv \, dl = \int_0^b \int_0^c \sqrt{(u^2 + v^2 - 2uv \cos A)} \, du \, dv \\ = \frac{1}{2} \int_0^b \left[ u^2 \cos A + (c - u \cos A) \sqrt{(u^2 + c^2 - 2uc \cos A)} \right. \\ \left. + u^2 \sin^2 A \log \left( \frac{c - u \cos A + \sqrt{(u^2 + c^2 - 2uc \cos A)}}{n(1 - \cos A)} \right) \right] du \\ = \frac{1}{6} (b^3 + c^3 - a^3) \cos A + \frac{1}{3} abc \sin^2 A + \frac{1}{6} b^3 \sin^2 A \log(\cot \frac{1}{2} A \cot \frac{1}{2} B) \\ + \frac{1}{6} c^3 \sin^2 A \log(\cot \frac{1}{2} A \cot \frac{1}{2} C). \\ \therefore D = \frac{1}{6} [(b^3 + c^3 - a^3) \cos A + (a^3 + c^3 - b^3) \cos B + (a^3 + b^3 - c^3) \cos C + 2abc(\sin^2 A \\ + \sin^2 B + \sin^2 C) + b^3 \sin^2 A \log(\cot \frac{1}{2} A \cot \frac{1}{2} B) + a^3 \sin^2 B \log(\cot \frac{1}{2} A \cot \frac{1}{2} B) \\ + (c^3 \sin^2 A + a^3 \sin^2 C) \log(\cot \frac{1}{2} A \cot \frac{1}{2} C) \\ + (b^3 \sin^2 C + c^3 \sin^2 B) \log(\cot \frac{1}{2} B \cot \frac{1}{2} C)].$$

If the line is terminated by two of the sides

$$M_1 = \frac{\int_0^b \int_0^c l_1 \, du \, dv + \int_0^a \int_0^c l_2 \, dw \, dx + \int_0^a \int_0^b l_3 \, dy \, dz}{\int_0^b \int_0^c du \, dv + \int_0^a \int_0^c dw \, dx + \int_0^a \int_0^b dy \, dz} = \frac{D}{bc + ac + ab}.$$

Corollary. If  $a=b=c$ ,  $N=\frac{5}{8}a^4$ ,  $D=a^3(1+\log 3)$ .

$$\therefore M = \frac{5a}{8(1+\log 3)}. \quad M_1 = \frac{1}{3}a(1+\log 3).$$

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**CALCULUS.**

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207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If  $K$  represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{K d\kappa}{1+\kappa} = \frac{1}{4}\pi^2.$$

Solution by S. A. COREY, Hiteman, Iowa.

As  $K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}$ , the definite integral may be written

$$\int_0^1 \int_0^1 \frac{dx d\kappa}{(1+\kappa)\sqrt{(1-x^2)(1-\kappa^2 x^2)}},$$

$$\text{or } \int_0^1 \frac{1}{(1-x^2)^{\frac{1}{2}}} \left[ \int_0^1 \frac{d\kappa}{(1+\kappa)(1-\kappa^2 x^2)^{\frac{1}{2}}} \right] dx \dots \dots (1).$$

$$\text{If } r=\kappa+1, \int_0^1 \frac{d\kappa}{(1+\kappa)\sqrt{(1-\kappa^2 x^2)}} = \int_1^2 \frac{dr}{r\sqrt{X}} =$$

$$- \left[ \frac{1}{\sqrt{a}} \log \left( \frac{\sqrt{X} + \sqrt{a}}{x} + \frac{b}{2\sqrt{a}} \right) \right]_1^2 = - \frac{1}{\sqrt{a}} \log \left( \frac{1}{1+\sqrt{a}} \right) = \frac{1}{\sqrt{a}} \log(1+\sqrt{a}),$$

in which  $a=(1+x^2)$ ,  $b=2x^2$ , and  $X=(1-x^2)+2x^2r-x^2r^2$ .

The definite integral (1), after substituting, becomes,

$$\int_0^1 \frac{\log[1+\sqrt{(1-x^2)}]}{(1-x^2)} dx, \text{ or, if } x = \frac{1-z^2}{1+z^2}, \quad \int_0^1 \frac{1}{z} \cdot \log \left( \frac{(1+z)^2}{1+z^2} \right) dz$$

$$= 2 \int_0^1 \frac{\log(1+z)}{z} dz - \int_0^1 \frac{\log(1+z^2)}{z} dz = 2 \cdot \frac{\pi^2}{6} - \frac{\pi^2}{2 \cdot 6} = \frac{\pi^4}{4}.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$(a^2+x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0 = \frac{d}{dx} \left[ (a^2+x^2) \frac{dy}{dx} \right].$$